Remarks for Tutorials on H2. 22 Sep.

1. Prop(*): To show that $\sup A \leq b$, $\left(\begin{array}{l}\emptyset \neq A \subseteq \mathbb{R},\ A \text{ bounded} \\ \text{above},\ b \in \mathbb{R}\end{array}\right)$

it suffices to show that

$\forall a \in A.\ a \leq b.$

Alternative proof for the Prop which may be easier:

Suppose we have: $\forall a \in A,\ a \leq b.$

Then b is an upper bound of A by definition.

Since by definition, $\sup A$ is the least upper bound of A,

we have $\sup A \leq b.$

2. Ex: Show that $\sup A = b$, where $A := [a, b)$.

In the proof of (ii"), we assumed that "$\sup A > b$" and aimed for contradiction.

We set $\varepsilon := \frac{\sup A - b}{2} > 0$, but we got into trouble that it may happen that

$$b - \varepsilon < a.$$

s.t. $b - \varepsilon \notin A.$

To fix the problem, we impose a condition on ε when we chose it: We redefine ε as:

$$\varepsilon := \min\left\{ \frac{\sup A - b}{2},\ b - a \right\} > 0$$

Then it is forced to be true that $b - \varepsilon \geq b - (b - a) = a.$

3. In the proof of " $\sup A + \sup B \le \sup(A+B)$ ", we use Prop $(*)$ twice. See if you can do it.

Alternatively, you may use (ii") directly.

$(l = \sup A)$

4. (ii") ~~For~~ $\forall \varepsilon > 0$, $\exists s \in A$ s.t. $s + \varepsilon > l$.

~~Alt~~ To check (ii"), it suffices to check for small ε $0 < \varepsilon \le \varepsilon_0$ for some fixed ε_0:

(ii"') is equivalent to (ii").

(ii"'): There is some $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \le \varepsilon_0$, $\exists s \in A$, s.t. $s + \varepsilon > l$.

Proof: (ii") $\Rightarrow$ (ii"') trivial, say, take $\varepsilon_0 = 1$.

(ii"') $\Rightarrow$ (ii") Let $\varepsilon > 0$. If $\varepsilon \le \varepsilon_0$, then by (ii"'), we are done; if $\varepsilon > \varepsilon_0$, then by (ii"'), for this ε_0,

$\exists s$ (depending on the same ε_0) $\in A$ s.t.

$$s + \varepsilon_0 > l.$$

Then $s + \varepsilon > s + \varepsilon_0 > l$.

Hence (ii") is true.

5. <u>WARNING:</u>

In case you are given a question asking you to <u>PROVE BY DEFINITION</u>, <u>no</u> properties and theorems in the tutorials can be directly used !!!